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Mathematical Modelling of Forced Convection in a Porous Medium for a General Geometry: Solution of Thermal Energy Equation Via Taylor's Series with Ying Buzu Algorithms

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The effects of thermal dispersion on forced convection inside a porous-saturated pipe were studied. The pipe wall is considered to maintain a constant and balanced heat flux. This model is based on a nonlinear equation containing a nonlinear term related to viscous dissipation, heat source terms and axial conduction. The steady-state thermal energy equation is solved using Taylor's series method coupled with the Ying Buzu algorithm. A numerical solution is also provided that is valid for the wide range of thermal dispersion conductivity. Furthermore, the outcome results based on present investigation are in good agreement with the literature.

Keywords: nonlinear equations, forced convection, porous-saturated duct, Taylor's series method, Ying Buzu algorithm.

1. INTRODUCTION

Convection in solid matrix porous media with an interrelated voids is a well-developed topic for researchers due to its significance to various engineering applications such as geothermal systems, sub-surface fire control, coal and grain storage, and energy recovery in high-temperature furnaces [1–3]. Porous heat exchangers have been identified for prospective applications in solar thermal plants [4], cooling towers [5], electronic cooling [6], diesel engines [7], and thermal storage systems [8]. The effects of thermal dispersion on convection in porous media have been investigated [9]. Also, the effects of thermal dispersion on forced/free convection occur in thermogalvanic cells and electrochemical sensors

[10], cooling of a lithium-ion battery[11], electrochemical cells and its application to redox flow batteries[12]and continuous electrochemical heat engines[13].

Convective diffusion processes are very important in electrochemistry, especially when these processes occur in porous media. Hence in the present paper, we examine the situation of forced convection in the absence of diffusive mass transport in a porous medium to obtain further insight into this complex problem. However, no analytical solution with inertia and convective term effects incorporated in the fully developed momentum transfer equation in porous media has been described in the literature.

However, an analytical solution to the temperature distribution can be obtained using functions of the log, hyperbolic, polylogarithms, and elliptic with imaginary arguments [9, 14], which are however too complicated in engineering applications. For non-Darcy flow issues, numerical simulations using Matlab have been employed primarily in the literature [15-18]. Hunt et al. [19] and Lemos et al. 20] investigated non-Darcian forced convection flow and heat transfer in high-porosity fibrous media and compared their results with experimental data.. The influence of thermal dispersion in periodic porous media consisting of an inline array of rectangular rods was computationally studied by Ozgumus and Mobedi [21,22].

Very recently, Abbasbandy et al. [23] obtained an exact analytical solution of some nonlinear equations arising from heat transfer problems which were expressed in complicated implicit form. Hooman et al, [24] presented an asymptotic solution of this problem for limiting cases. In this paper we have obtained an analytical expression for velocity filtration using the Taylors series method. The analytical results are compared with numerical results and satisfactory agreement is observed.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

A unidirectional flow exists in the x*- direction inside a channel with impermeable walls at y*= H for the fully developed parallel flow across a horizontal channel, as shown in Fig. 1. The heat flux at the tube wall remains constant at q''. The momentum equation of Brinkman–Forchheimer is

$$\mu_{eff}\left(\frac{d^2u^*(r^*)}{dr^{*2}}+\frac{n}{r^*}\frac{du^*(r^*)}{dr^*}\right)-\frac{\mu}{k}u^*(r)-\frac{C_F\rho u^*(r^*)^2}{\sqrt{k}}+G=0 \quad (1)$$

÷

where μ_{eff} is an effective viscosity, μ represents fluid viscosity, k denotes permeability, ρ stands for fluid density, C_F represents the inertial coefficient, and G denotes the negative of the applied pressure gradient. For planar, cylindrical, and spherical tubes, n=0,1,2 is also valid. The dimensionless variables are defined as follows:

$$x=\frac{x*}{PeR}, r=\frac{r*}{R}, u=\frac{\mu u*}{GR^2} \quad (2)$$

where $P_e=\frac{\rho c_p R\, U}{k}$ is Peclet number. Now, the Eqn. (1) can be written in dimensionless form as follows:

$$M\left(\frac{d^2u(r)}{dr^2}+\frac{n}{r}\frac{du(r)}{dr}\right)-\frac{u(r)}{D_a}-\frac{MFu(r)^2}{\sqrt{D_a}}+1=0 \quad (3)$$

where the viscosity ratio (M), the Darcy number (D_a), and Forchheimer number (F) are defined by

$$M=\frac{\mu_{eff}}{\mu}, \qquad D_a=\frac{k}{R^2}, \qquad F=\frac{C_F\rho GR^3}{\mu_{eff}\mu} \quad (4)$$

The Eqn. (3) can be described as

$$\frac{d^2u(r)}{dr^2}+\frac{n}{r}\frac{du(r)}{dr}-s^2u(r)-Fsu(r)^2+\frac{1}{M}=0 \qquad (5)$$

where porous media shape parameter s is defined as

$$s=\frac{1}{\sqrt{MD_a}} \qquad (6)$$

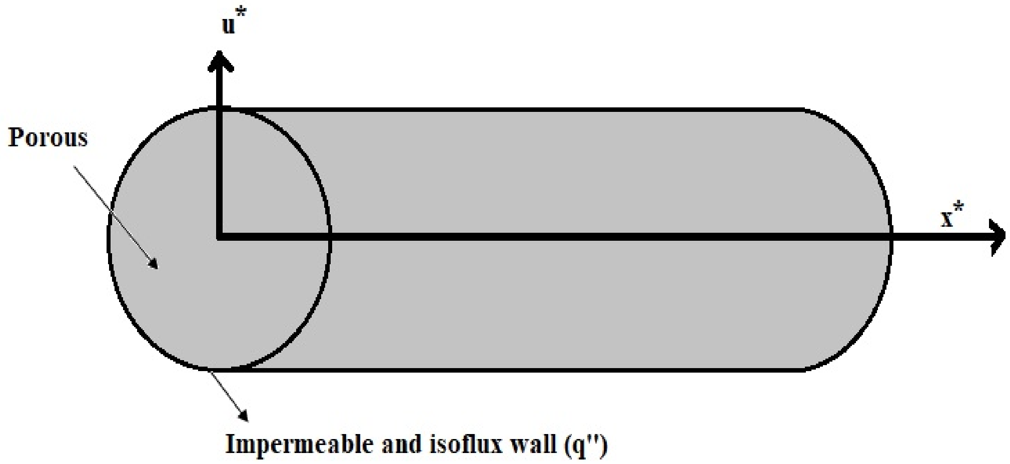


Figure 1. Systematic diagram of unidirectional flow in porous-saturated pipe.

Non-slip and symmetry boundary condition for the Eqn. (5) are given by

$$u'(0)=0,\ u(1)=0 \qquad (7)$$

3. ANALYTICAL SOLUTION OF NONLINEAR PROBLEM

3.1 Taylor's series method

This section describes the solution of the nonlinear boundary value problem (5) using the Taylors series and Ying Buzu algorithm. Taylor's series method (TSM) [25-33] produces a semi-analytical solution in the form of a fast converging series that does not involve linearization. The analytical expression of velocity in the porous saturated duct's overall shape is provided by

$$u(r)=\sum_{i=0}^{6}\frac{d^iu}{dr^i}\frac{(r)^i}{i!}\bigg|_{y=0}=u(0)+\frac{(r)}{1!}\frac{du}{dr}\bigg|_{r=0}+\frac{(r)^2}{2!}\frac{d^2u}{dr^2}\bigg|_{r=0}+\frac{(r)^3}{3!}\frac{d^3u}{dr^3}\bigg|_{r=0}\cdots$$

$$=u(0)+u_1(0)\frac{r^2}{2!}+u_2(0)\frac{r^4}{4!}+u_3(0)\frac{r^6}{6!} \qquad (8)$$

The unknown parameter $u(0)$ can be obtained from the following equation.

$$u(0)+u_1(0)\frac{1^2}{2!}+u_2(0)\frac{1^4}{4!}+u_3(0)\frac{1^6}{6!}=0 \qquad (9)$$

where

$$u_1(0)=\frac{s^2\,u(0)+F\,s\,u(0)^2-\left({}^1/_M\right)}{(n+1)}$$

$$u_2(0)=\frac{3\,s\,(2\,F\,u(0)+s)\left(s^2\,u(0)+F\,s\,u(0)^2-\left({}^1/_M\right)\right)}{(n+1)(n+3)} \qquad (10)$$

$$u_3(0)=\frac{\left\{\begin{matrix}15s\,(2\,F\,u(0)+s)\left(s^2\,u(0)+F\,s\,u(0)^2-\left({}^1/_M\right)\right)\\ \left[(3n+5)(2Fs\,u(0))(Fu(0)+s)+s^3(n+1)-\frac{2F}{M}(n+3)\right]\end{matrix}\right\}}{(n+1)^2(n+3)(n+5)}$$

The boundary condition $u(1)=0$ can be used to generate equation (9). The Ying Buzu algorithm, detailed in the following section, can be used to find the unknown parameter $u(0)$. The regular false method (Appendix A) and the secant algorithm (Appendix B) were also used to determine this parameter. Ying Buzu algorithm method presented in this paper offer an extremely fast convergent result.

3.2. The Ying Buzu algorithm

A brief introduction to the Ying Buzu algorithm is given in Ref [34], and this algorithm is applied to solve nonlinear oscillators [35-41] and fractal vibration systems [42,43]. Recently He used an ancient Chinese algorithm [44-46] to solve the nonlinear equations. The basic concept of this algorithm is given below. Considering the nonlinear differential equation:

$$u''(r) + F(u(r)) = 0 \tag{11}$$

The boundary conditions are

$$u'(a) = \alpha \tag{12}$$

$$u(b) = \beta \tag{13}$$

where *a, b* are the terminal points of the boundary [*a, b*] and α, β are the given real numbers.

Here $u(a)$ (This is equal $u(0)$ in the Eqn. (9)) is unknown parameter. We can obtain this parameter using Ying Buzu algorithm as follows: We can assume the initial guess of $u(a)$ as

$$u_1(a) = a_1, u_2(a) = a_2 \tag{14}$$

where a_1 and a_2 are taken as the positive values and less than b. Using the initial conditions given in Eqn. (12) and Eqn. (13), we can obtain the terminal values:

$$u_1(b) = \beta_1, u_2(b) = \beta_2 \tag{15}$$

According to the Ying Buzu algorithm [36-41] and fractal vibration systems [42,43]. Recently He used an ancient Chinese algorithm [44-46] and [42,43] the initial guess can be updated as

$$u(a)_{estimate} = a_3 = \frac{u_1(a)(u(b) - u_2(b)) - u_2(a)(u(b) - u_1(b))}{(u(b) - u_2(b)) - (u(b) - u_1(b))}$$
$$= \frac{a_1(\beta-\beta_2)-a_2(\beta-\beta_1)}{(\beta-\beta_2)-(\beta-\beta_1)} \tag{16}$$

For these experimental value of parameter $s = 0.5, F = 1$ and $M = 1,$ we assume the initial guess for the problem (Eqn. (5)) as follows:

$$u_1(0) = 0.4, \ u_2(0) = 0.5 \tag{17}$$

We get the following result from the Eqn.(8)

$$u_1(r) = 0.4 - 0.41r^2 - 0.02221r^4 - 0.002320\, r^6 + 0.0001895r^8 + .. \tag{18}$$

$$u_2(r) = 0.5 - 0.375r^2 - 0.02344\, r^4 - 0.001758r^6 + 0.0001805r^8 + .. \tag{19}$$

From the above equation, we get $u_1(1) = -0.02969$ and $u_2(1) = 0.1035$ respectively. Using Eqn. (17)

$$u_{3\,(estimate)}(0) = \frac{u_1(0)(u(1) - u_2(1)) - u_2(0)(u(1) - u_1(1))}{(u(1) - u_2(1)) - (u(1) - u_1(1))}$$
$$= \frac{0.4(0-0.1035)-0.5(0+0.02969)}{(0-0.1035)-(0+0.02969)} = 0.4227 \tag{20}$$

Now using Eqn. (18), we get

$u_3(1) = 0.0004$ (21)

This result is very close to $u(1) = 0$ with a relative error of 0.04%. Hence using the first iteration we get

$$u(r) = 0.4227 - 0.4025\, r^2 - 0.02256\, r^4 - 0.002194\, r^6 + 0.0001885\, r^8 \ldots \quad (22)$$

We can continue the iteration process to obtain a higher accuracy.

3.3 Previous result for planar geometry

Abbasbandy et al. [23] obtained the dimensionless filtration velocity for planar geometry in implicit form as follows:

$$r = G(u; s, F, M, u(0)) = \int_u^{u(0)} \frac{d\theta}{\sqrt{s^2u^2 + \frac{2}{3}Fsu^2 - \frac{2}{M}u - s^2u(0)^2 - \frac{2}{3}Fsu(0)^3 + \frac{2}{M}u(0)}} \quad (23)$$

For instance

$$G(u\!:1{,}1{,}1, u(0)) = \frac{\sqrt{\frac{(A_1)^2}{3+6u(0)+A_1}}}{A_2A_3}\left(\begin{array}{c}\sqrt{\frac{3+6u(0)+A_1}{A_1}} - \frac{-3-6u(0)+A_1}{A_1}\, Eliptic\ F\left[\sqrt{\frac{3+6u(0)+A_1}{2A_1}}, \sqrt{\frac{2A_1}{3+6u(0)+A_1}}\right] A_2 \\ 3\sqrt{\frac{3+2u(0)+A_1+4u}{A_1}}\sqrt{u_0 - u}\sqrt{\frac{-3-2u(0)+A_1+4u}{A_1}} \times \\ Eliptic\ F\left(\sqrt{\frac{3+2u(0)+A_1+4u}{A_1}}, \sqrt{\frac{2A_1}{3+6u(0)+A_1}}\right) A_3\end{array}\right) \quad (24)$$

where

$$A_1 = \sqrt{57 - 12u(0) - 12u(0)^2}$$

$$A_2 = \sqrt{19u^2 - 18u + 6u^3 - 6\,u(0)^3 + 18u(0) - 9\,u(0)^2}, \quad A_3 = \sqrt{-2u(0) + 2 - 2\,u(0)^2}$$

And Elliptic *F* is the incomplete elliptic integral of the first kind defined as follows:

$$Eliptic\ F\ (z, k) = \int_0^1 \frac{d\tau}{\sqrt{1-\tau^2}\sqrt{1-k^2\tau^2}} \quad (25)$$

4. DISCUSSION

Eqn. (8) represent the new analytical expression for the velocity in porous media for all experimental values of parameters such as viscosity ratio (M), shape factor (s) and drag coefficient or Forchheimer number (F). Our obtained analytical result for the filtration velocity is very simple and easily computable compared to the previous result reported by Abbasbandy et al. [23] and presented in Eqn. (23).

4.1 validation of analytical methods.

Method validation is an essential part of developing reference methods. The value of $u(0)$, which is obtained from ancient Chinese algorithm used in the present paper, is compared with the previous result reported by Abbasbandy et al. [23] and simulation results for various values of the parameters F, M and s are outlined in Tables 1-2. A satisfactory agreement is noted between the present work and those presented previously by Abbasbandy et al [23].

Table 1. Comparison of simulations values of $u(0)$ with previous result [23] and this work (Taylor's series with ancient algorithms) for various values of Forchheimer number (F) when $M = s = 1$

F	$u(0)$			Error % TSM &ACA	Error % Abbasbandy [23]
	Numerical	TSM &ACA This work (8)	Abbasbandy [23] using (24)		
0	0.3519	0.3519	0.3519	0.00	0.00
1	0.3239	0.3239	0.3238	0.03	0.03
2	0.3026	0.3026	0.3026	0.00	0.00
3	0.2857	0.2857	0.2857	0.00	0.00
4	0.2717	0.2717	0.2717	0.00	0.00
5	0.2598	0.2598	0.2598	0.00	0.00
6	0.2496	0.2497	0.2494	0.04	0.08
7	0.2406	0.2407	0.2406	0.04	0.00
8	0.2326	0.2327	0.2326	0.04	0.00
9	0.2254	0.2255	0.2254	0.04	0.00
10	0.2190	0.2191	0.2190	0.05	0.00
	Average % error			0.02	0.01

Table 2. Comparison of simulation values of $u(0)$ with previous result [19] and this wok (Taylor's series with ancient Chinese algorithms) for various values of shaped parameter (s) when $M = F = 1$

s	$u(0)$			Error % TSM&ACA	Error % Abbasbandy [23]
	Numerical	TSM & ACA This work (8)	Abbasbandy [23] using (24)		
0	0.5000	0.5000	0.5000	0.00	0.00
0.5	0.4227	0.4227	0.4227	0.00	0.00
1.0	0.3239	0.3239	0.3238	0.00	0.03
1.5	0.2384	0.2384	0.2384	0.00	0.00
2.0	0.1744	0.1745	0.1744	0.06	0.00
2.5	0.1292	0.1292	0.1292	0.00	0.00
3.0	0.0976	0.0976	0.0976	0.00	0.00
3.5	0.0753	0.0753	0.0753	0.00	0.00
4.0	0.0594	0.0594	0.0594	0.00	0.00
4.5	0.0478	0.0478	0.0478	0.00	0.00
5.0	0.0392	0.0392	0.0392	0.00	0.00
	Average % error			0.01	0.00

Table 3 shows a comparison of u(0) with simulation results and our results from the Ancient Chinese algorithm, the Regular- Falsi algorithm, and the Secant technique for various experimental values parameters.

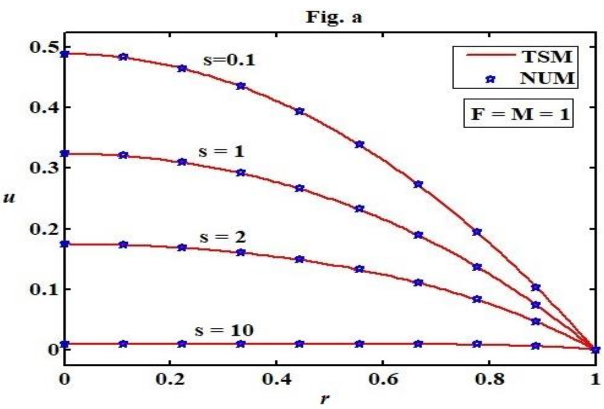
Table 3. Comparison of $u(0)$ with simulation results and our result obtained from Ancient Chinese algorithm, Regular- Falsi algorithm & Secant method for various experimental values parameters.

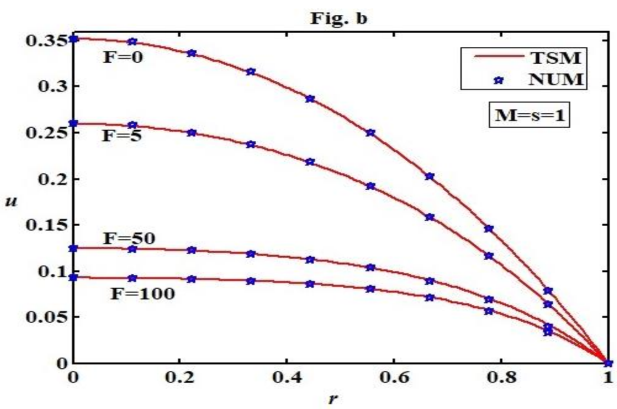
F	s	M	Taylor's series and Algorithm			Numerical value u(0)	Err %
			The initial guess of $u(0)$	Corresponding terminal values $u(1)$	*Estimated value of u(0)		
0	1	1	$u(0) = 0.3$ and 0.4	$u(1) = -0.0802, u(1) = 0.0742$	0.3519	0.3519	0.00
1	1	1	$u(0) = 0.3$ and 0.35	$u(1) = -0.0443, u(1) = 0.0494$	0.3236	0.3236	0.09
2	1	1	$u(0) = 0.29$ and 0.31	$u(1) = -0.2073, u(1) = 0.0162$	0.3026	0.3026	0.00
3	1	1	$u(0) = 0.25$ and 0.3	$u(1) = -0.0849, u(1) = 0.0359$	0.2851	0.2851	0.21
4	1	1	$u(0) = 0.25$ and 0.3	$u(1) = -0.0581, u(1) = 0.0813$	0.2708	0.2708	0.33
5	1	1	$u(0) = 0.23$ and 0.27	$u(1) = -0.0859, u(1) = 0.0313$	0.2593	0.2593	0.19
						Average	**0.14**

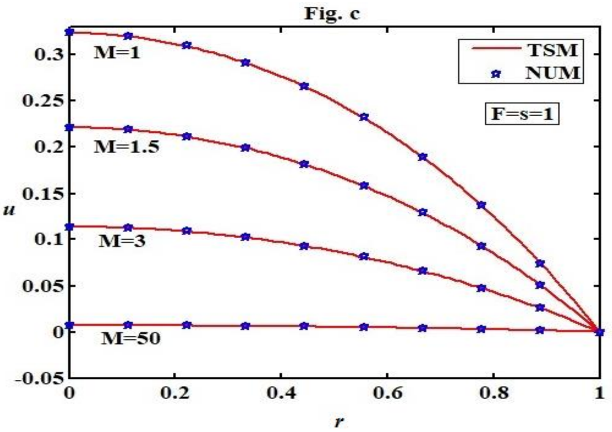
***** Ancient Chinese algorithm, Regular- Falsi algorithm & Secant method

4.2 Effect of the parameters on velocity field.

The velocity gradient at the near-wall region depends upon the parameters F, M and s. The velocity field is often dominated by buoyancy forces when the forced flow is weak, but the acceleration generated by buoyancy forces deflects the main flow toward the heat source. Figures 2(a) to 2(c) illustrate the behaviour of the velocity profiles for different values of the F, M and s. The analytical results show that decreasing the values of F, M and s results in an increasing velocity. The velocity profile for $s \leq 0.1$ *or* $M \leq 1$ is closer to that of clear flow and represents a parabola. In contrast, for $s \geq 10$ *or* $M \geq 50$, the velocity profile is flattened and approaches the Darcy flow, as seen in Figures 2a and 2d. From Figure 2(d), it is observed that an increase in the value of n results in a decrease in velocity field. When F=0 and s is large the velocity tends to Darcy flow.







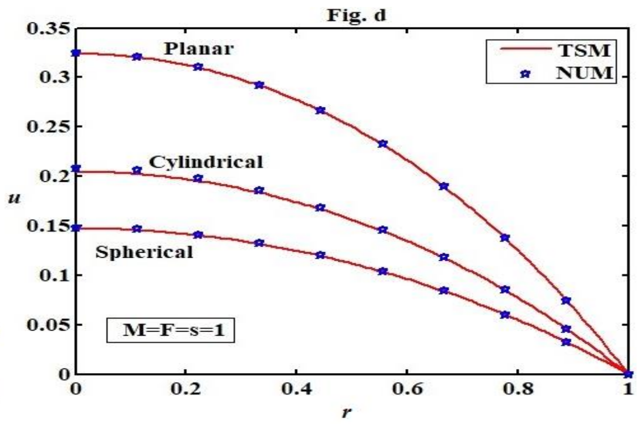


Figure 2. Plot of the analytical expression for $u(r)$ when M = F = 1 for different porous media shaped parameter.

5. CONCLUSIONS

The effects of thermal dispersion on fully developed forced convection in a porous-saturated pipe were investigated**.** A nonlinear boundary value problem arising from forced convection in a porous-saturated conduit is solved using the Taylors series and the Ying Buzu algorithm. The velocity profile is expressed in terms of a simple power series. The effects of the parameter on the velocity field are also discussed. The temperature and Nusselt number are calculated as a function of the critical factors, including the thermal dispersion coefficient, using this analytical expression of velocity. The velocity profile is investigated using numerical techniques (Matlab). Analytical results are compared to simulation results, and there is satisfactory agreement.

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K. Lakshmi Narayanan: Data curation, Software, Formal analysis, Writing- original draft, R. Shanthi: Data curation, Software, Formal analysis, Ramu Usha Rani: Formal analysis, Writing- original draft, Visualization, Formal analysis, Investigation, Michal E. G. Lyons: Investigation, Validation, Lakshamanan Rajendran: Formal analysis, Investigation, Writing - review & editing, Conceptualization, Methodology, Resources, Project administration, Supervision.

DECLARATION OF COMPETING INTEREST
The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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NOMENCLATURE:

Symbols	Meaning	Units
C	Constant	None
C_F	Inertial coefficient	None
c_p	Specific heat at constant pressure	$J\,kg^{-1}K^{-1}$
D_a	Darcy number, $\frac{K}{H^2}$	None
F	Forchheimer number	None
G	Negative of the applied pressure gradient	$Pa\,m^{-1}$
k	Effective thermal conductivity	$W\,m^{-1}K^{-1}$
k_f	Fluid thermal conductivity	$W\,m^{-1}K^{-1}$
K	permeability	m^2
M	$\frac{\mu_{eff}}{\mu}$ Dimensionless parameter	None
Nu	Nusselt number	None
O	Symbol for order of magnitude	None
P_e	Peclet number	None
P_r	Prandtl number	None
P_{r_f}	Fluid Prandtl number	None
q''	Wall heat flux	$W\,m^{-2}$
R	Tube radius	m
Re	Pore- Reynolds number	None
s	Porous media shape parameter, $(MDa)^{-\frac{1}{2}}$	None
T^*	Temperature	K
T_m	Bulk mean temperature	K
T_w	Downstream wall temperature	K
u	Dimensionless filtration velocity $\frac{\mu u^*}{GH^2}$	None
u^*	Filtration velocity	m
$\hat{u}$	u^*/U	None
U	Mean velocity	$m\,s^{-1}$
x^*	Longitudinal coordinate	m
x	Dimensionless longitudinal coordinate, x^*/PeH	None
r^*	Radial coordinate	m
r	Dimensionless radial coordinate r^*/R	None

Greek Symbols

η	Stretched variable	None

γ	Dimensionless number (γ=0.025)	None
θ	Effective viscosity	$Ns\, m^{-2}$
μ	Fluid viscosity	$Ns\, m^{-2}$
μ_{eff}	Effective viscosity	$Ns\, m^{-2}$
ρ	Fluid density	$kg\, m^{-3}$

Subscript

Prime	Differentiation with respect to *y*	None

APPENDIX-A: REGULAR-FALSI ALGORITHM

This method is also known [47-50] as False position method. Consider the equation $u(r) = 0$ and assume that $u(a)$ and $u(b)$ have opposite signs. Also let $a < b$ by taking the boundary [*a*,b] =[0,1]. The general formula in regular-falsi method is

$$r_1 = \frac{au(b)-bu(a)}{u(b)-u(a)} \qquad \text{(A1)}$$

For example, take $a = 0.2$, $b = 0.3$, and $u(a)u(b) < 0$.When $a = 0.2$ we get from Eqn. (2.1)

$$u(r) = 0.2 - 0.2325r^2 - 0.006539r^4 - 0.0006690r^6 + \cdots \qquad \text{(A2)}$$

When $b = 0.2$ we get from Eqn. (8)

$$u(r) = 0.3 - 0.2200r^2 - 0.007563\, r^4 - 0.0005567r^6 + \cdots \qquad \text{(A3)}$$

Also $u(a) = -0.03837$ and $u(b) = 0.07299$

$$r_1 = \frac{au(b)-bu(a)}{u(b)-u(a)} = \frac{0.2(0.07299)-0.3(-0.03837)}{(0.07299)-(-0.03837)} = 0.2344 \qquad \text{(A4)}$$

The regular- falsi process, using Eqn. (8) results in

$$u(1) = -0.0003 \qquad \text{(A5)}$$

Which derivates the exact value of $u(1) = 0$ with a relative error of 0.03%. Now $u(r)$ becomes

$$u(r) = 0.2324 - 0.1784\, r^2 - 0.01633\, r^4 - 0.0002195\, r^6 \ldots \qquad \text{(A6)}$$

APPENDIX-B: SECANT METHOD

The secant line is defined using two points on the graph of $u(r)$, as opposed to a tangent line that requires information at only one point on the graph, it is necessary to choose two initial iterates r_0, and r_1 [51-54]. Then the next iterate r_2 is then obtained by computing the *r*-value at which the secant line passing through the points $(r_0, u(r_0))$ and $(r_1, u(r_1))$ has a r-coordinate of zero. However, in many other cases, it is expensive to compute the first derivative, and the above methods are still restricted in practical applications. The well-known secant method is given by

$$r_{n+1} = r_n - \frac{r_n - r_{n-1}}{u(r_n)-u(r_{n-1})} u(r_n) \qquad \text{(B1)}$$

When *n*=0, this result becomes

$$r_2 = r_1 - \frac{u(r_1)(r_1-r_0)}{u(r_1)-u(r_0)} \qquad \text{(B2)}$$

For example, assume that $r_0 = 0.2, r_1 = 0.3,$ and $u(r_0)u(r_1) < 0.$ When $r_0 = 0.2$ we get from Eqn. (8)

$$u(r) = 0.2 - 0.155r^2 - 0.003588r^4 - 0.0002486\, r^6 + \cdots \quad \text{(B3)}$$

Consequently, we get from Eqn .(8) when u(0)=0.3

$$u(r) = 0.3 - 0.14667r^2 - 0.004033\, r^4 - 0.0002033\, r^6 + \cdots \quad \text{(B4)}$$

We get $u(r_1) = -0.06421$ for $u(0) = 0.2$, $u(r_2) = 0.04176$ for $u(0) = 0.3$

Now the estimated value of r become

$$r_2 = r_1 - \frac{r_1 - r_0}{u(r_1) - u(r_0)} u(r_1) = 0.2 - \frac{0.04176\ (0.2 - 0.1)}{0.04176 - (-0.06421)} = 0.1606 \quad \text{(B5)}$$

The Secant process using Eqn. (B5) results in

$$u(1) = -0.0002 \quad \text{(B6)}$$

This value is very close the exact value of $u(1) = 0$ with a relative error of 0.02%.

$$u(r) = 0.1606 - 0.1356\, r^2 - 0.008958\, r^4 - 0.0001560\, r^6 + \cdots$$

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